Q: $\int \frac{1+x^3+x^4+x^5}{1+x^2+x^4+x^6}dx = ?$

A:

First, $1+x^2+x^4+x^6 = (1+x^2)(1+x^4) = (1+x^2)(1+\sqrt{2}x+x^2)(1-\sqrt{2}x+x^2)$

(i) A fool will solve the problem like this:

$$\int \frac{1+x^3+x^4+x^5}{1+x^2+x^4+x^6}dx = \int \frac{1+x^3+x^4+x^5}{(1+\mathrm{x}^2)(1+\sqrt{2}\mathrm{x}+\mathrm{x}^2)(1-\sqrt{2}\mathrm{x}+\mathrm{x}^2)}dx$$

$$= \int\left(\frac{Ax+B}{1+\mathrm{x}^2}+\frac{Cx+D}{1+\sqrt{2}\mathrm{x}+\mathrm{x}^2}+\frac{Ex+F}{1-\sqrt{2}\mathrm{x}+\mathrm{x}^2}\right)dx$$

And then get the answer to the indefinite integral by evaluating A, B, C, D, E, and F. This is nuts! I'm showing you a much cleverer one.

(ii) A smart guy does this:

$\because\ 1+x^3+x^4+x^5 = (1+x^4) + x^3(1+x^2)$

$$\therefore \int \frac{1+x^3+x^4+x^5}{1+x^2+x^4+x^6}dx = \int \frac{1+x^3+x^4+x^5}{(1+\mathrm{x}^2)(1+\mathrm{x}^4)}dx$$

$$= \int\left(\frac{1}{1+\mathrm{x}^2}+\frac{x^3}{1+\mathrm{x}^4}\right)dx$$

$$= \tan^{-1}x + \frac{1}{4}\ln(1+x^4) + \mathrm{C}$$